The Irrationality of π

Sri Pranav K.

October 2022

Defining the rational (and irrational) numbers

Let's define what it means for a number to be "rational" or "irrational." You only need to know the definition of a rational number, because irrational just means "not rational."

Definition: Rational Number

A number x is said to be **rational** if (and only if) $x = \frac{p}{q}$ for integers p and $q \neq 0$. If no such integers exist, x is said to be **irrational**.

We care about rational and irrational numbers because a lot of things go wrong in math without the existence of irrational numbers. For example, we will show later that $\log_{10}(3)$ is irrational.

Is the product of irrational numbers always irrational? What about the sum?

Guiding question 2

If a number's decimal representation terminates after a finite amount of digits, is it rational? Explain.

$\log_{10}(3)$ is irrational

Suppose for a contradiction that $\log_{10}(3)$ is rational. In particular, for integers p and q, $\log_{10}(3) = \frac{p}{q}$. This means

$$10^{\frac{p}{q}} = 3 \iff 10^p = 3^q.$$

However, since p and q are integers, this can't be true since 10 is even and 3 is odd. An odd number raised to any positive integer power remains odd, and a similar statement can be made with even numbers. Hence we have reached a contradiction and $\log_{10}(3)$ must be irrational.

Why is π irrational? (1/6)

This proof is due to Nicolas Bourbaki, a group of French mathematicians. Its presentation has been modified for Calc BC.

First we will study properties of the following function, which is defined for each integer $n \geq 0$:

$$g_n(x) = \frac{x^n(\pi - x)^n}{n!}$$

Please write this down, we will need it throughout the proof.

Why is π irrational? (2/6)

Taking a look at g_n, we see that if we expand the numerator $x^n(\pi - x)^n$ as a polynomial, each term contains cx^m, for $n \leq m \leq 2n$ and some constant c.

Using what we know about differentiating polynomials, this means that for $0 \leq k < n$,

$$g_n^{(k)}(0) = 0$$

since $0^k = 0$ for $k > 0$. More relevant to our proof is that $g_n^{(k)}(0)$ is an integer, since 0 is an integer.

Why is π irrational? (3/6)

Suppose, for a contradiction, that $\pi = \frac{p}{q}$ for positive integers p and q. Define $f_n(x) = q^n g_n(x)$, and now we can rewrite this function by substituting $\frac{p}{q}$ for π:

$$f_n(x) = q^n \frac{x^n(\pi - x)^n}{n!} = q^n \frac{x^n \left(\frac{p}{q} - x\right)^n}{n!} = \frac{x^n(p - qx)^n}{n!}.$$

For $n \leq k \leq 2n$, we see that the constant term of $f_n^{(k)}$ is of the form $\frac{ck!}{n!}$ for some integer c.

Since $k > n$, $f_n^{(k)}(0)$ is an integer because the non-constant terms vanish upon differentiation at 0 and the product/sum of integers is an integer.

Why is π irrational? (4/6)

It also follows from the chain rule that for $0 \le k \le 2n$,

$$g_n^{(k)}(x) = (-1)^k g_n^{(k)}(\pi - x).$$

So, we can conclude that $f_n^{(k)}(0)$ and $f_n^{(k)}(\pi)$ are integers for $0 \le k \le 2n$. Now, let's go a bit further and define

$$A_n := \int_0^\pi f_n(x)\sin(x)\,dx = q^n \int_0^\pi \frac{x^n(\pi - x)^n}{n!}\sin(x)\,dx.$$

Using the tabular method for repeated integration by parts (remember that f_n is a polynomial) and the continuity of the integrand, we see from the FTC that

$$A_n = \Big[-f_n(x)\cos(x)\Big]_{x=0}^{x=\pi} \pm \cdots \pm \int_0^\pi f_n^{(2n+1)}\sin(x)\,dx.$$

Why is π irrational? (5/6)

Because f_n is a polynomial of degree $2n$, $f_n^{(2n+1)}(x) = 0$ for all x. Thus the final term is zero. Since $f_n^{(k)}(x)$, $\sin(x)$, and $\cos(x)$ are integers for $x = 0$ and $x = \pi$, A_n is an integer for all n.

Here is where we will make our contradiction. Let's study the same integral using some properties we already know. First, we know that for $0 < x < \pi$,

$$\frac{x^n(\pi - x)^n}{n!}\sin(x) > 0.$$

Hence $A_n > 0$. Now consider the decreasing parabola $x(\pi - x) = x\pi - x^2$. Using our vertex formula, it follows that

$$x(\pi - x) \leq \frac{\pi^2}{4}.$$

Why is π irrational? (6/6)

The previous inequality leads to

$$q^n \frac{x^n(\pi - x)^n}{n!} \sin(x) \leq \left(\frac{q\pi^2}{4}\right)^n \frac{1}{n!}.$$

So, that means that for all n,

$$A_n < \int_0^\pi \left(\frac{q\pi^2}{4}\right)^n \frac{1}{n!}\, dx = \pi \left(\frac{q\pi^2}{4}\right)^n \frac{1}{n!}.$$

Then for sufficiently large n, $0 < A_n < 1$ (justified at the end). But this contradicts the fact that A_n is an integer for all n if π is rational.

Since there are no integers between 0 and 1, π is irrational. $\square$

Guiding question 3

Why did we choose $n!$ as the denominator of g_n (and f_n)?

Guiding question 4

We assumed p and q to be positive without loss of generality in the proof. Why does this proof also account for when p and q are assumed to be negative?

Extra: Showing that $A_n < 1$ when n is sufficiently large

Pick $N_0 > \frac{q\pi^2}{4}$ and $B > \max\{\pi N_0^{N_0}, 4N_0\}$. For $n > j = \left\lceil \frac{B\pi^2 q}{4} \right\rceil$,

$$\prod_{k=j}^{n} \left(\frac{q\pi^2}{4}\right)\frac{1}{k} < \left(\frac{q\pi^2}{4j}\right)^{n-j+1} < \frac{1}{B^{n-j+1}} < \frac{1}{B}$$

so that

$$1 > B\prod_{k=j}^{n}\left(\frac{q\pi^2}{4}\right)\frac{1}{k} > B\prod_{k=N_0+1}^{j-1}\left(\frac{q\pi^2}{4}\right)\frac{1}{k}\prod_{k=j}^{n}\left(\frac{q\pi^2}{4}\right)\frac{1}{k} > A_n$$

since the denominator of the products are equivalent to $n!$ excluding factors $1, \ldots, N_0$.